Breakdown of thermalization in finite one-dimensional systems

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*Finite-Temperature Non-Equilibrium Superfluid Systems*
Van Mildert College, Durham, UK
September 14-17, 2009
Outline

1 Introduction
   - Experiments and numerical simulations
     - Thermalization in quantum systems

2 Non-equilibrium dynamics in one-dimension
   - Time evolution vs exact time average
   - Statistical description after relaxation
   - Eigenstate thermalization hypothesis
   - Time fluctuations

3 Integrable systems
   - Generalized Gibbs ensemble

4 Summary
Experiments in the 1D regime

Effective one-dimensional $\delta$ potential
M. Olshanii, PRL 81, 938 (1998).

$$U_{1D}(x) = g_{1D} \delta(x)$$

where

$$g_{1D} = \frac{2\hbar a_s \omega_\perp}{1 - C a_s \sqrt{\frac{m \omega_\perp}{2\hbar}}}$$
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Girardeau ’60


$T_{1D}$: Average energy in the 1D tubes
$U_0$: Lattice depth ($\sim$ LASER intensity)

$$\gamma_{avg} = \frac{E_{int}}{E_{kin}}$$
Experiments in the 1D regime

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where

$$g_{1D} = \frac{2\hbar a_s \omega_{\perp}}{1 - C a_s \sqrt{\frac{m \omega_{\perp}}{2\hbar}}}$$

Lieb, Schulz, and Mattis '61

B. Paredes et al.,

$n(x)$: Density distribution

$n(p)$: Momentum distribution
Absence of thermalization in 1D?


Absence of thermalization in 1D?

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Experiment

Theory

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Breakdown of thermalization in 1D

September 17, 2009
A quantum Newton’s cradle?
Absence of thermalization in 1D

\[ \gamma = \frac{E_{\text{int}}}{E_{\text{kin}}} \]

- \( E_{\text{int}} \): Interaction energy
- \( E_{\text{kin}} \): Kinetic energy

If \( \gamma \gg 1 \) the system is in the strongly correlated Tonks-Girardeau regime.

If \( \gamma \ll 1 \) the system is in the weakly interacting regime.

Absence of thermalization in 1D numerical simulations

Hard-core bosons (integrable)
Absence of thermalization in 1D numerical simulations

Hard-core bosons (integrable)

Spinless fermions Hamiltonian

\[ H = -t \sum_j \left( c_{j+1}^\dagger c_j + h.c. \right) + V \sum_j n_j n_{j+1} + V_2 \sum_j n_j n_{j+2} \]


Momentum distribution function

\[ \langle n_k \rangle \]

Soft-core bosons
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4. Summary
Thermalization in quantum systems

If the initial state is not an eigenstate of $\hat{H}$

$$|\psi_I\rangle \neq |\Psi_\alpha\rangle \quad \text{where} \quad \hat{H}|\Psi_\alpha\rangle = E_\alpha |\Psi_\alpha\rangle \quad \text{and} \quad E_0 = \langle \psi_I | \hat{H} | \psi_I \rangle,$$

then a generic observable $A$ will evolve in time following

$$A(t) \equiv \langle \psi(t) | \hat{A} | \psi(t) \rangle \quad \text{where} \quad |\psi(t)\rangle = e^{-i\hat{H}t} |\psi_I\rangle.$$
Thermalization in quantum systems

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Will a generic $A$ in a generic system thermalize?

$$\overline{A(t)} = A(E_0) = A(T).$$
Thermalization in quantum systems

If the initial state is not an eigenstate of \( \hat{H} \)

\[
|\psi_I\rangle \neq |\Psi_\alpha\rangle \quad \text{where} \quad \hat{H}|\Psi_\alpha\rangle = E_\alpha|\Psi_\alpha\rangle \quad \text{and} \quad E_0 = \langle \psi_I | \hat{H} | \psi_I \rangle,
\]

then a generic observable \( A \) will evolve in time following

\[
A(t) \equiv \langle \psi(t) | \hat{A} | \psi(t) \rangle \quad \text{where} \quad |\psi(t)\rangle = e^{-i\hat{H}t} |\psi_I\rangle.
\]

Will a generic \( A \) in a generic system thermalize?

\[
\overline{A(t)} = A(E_0) = A(T).
\]

One can rewrite

\[
A(t) = \sum_{\alpha', \alpha} C_{\alpha'}^* C_\alpha e^{i(E_{\alpha'} - E_\alpha)t} A_{\alpha' \alpha} \quad \text{where} \quad |\psi_I\rangle = \sum_\alpha C_\alpha |\Psi_\alpha\rangle,
\]

and taking the infinite time average (diagonal ensemble)

\[
\overline{A(t)} = \sum_\alpha |C_\alpha|^2 A_{\alpha \alpha},
\]

which depends on the initial conditions through \( C_\alpha = \langle \Psi_\alpha | \psi_I \rangle \).
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4 Summary
Relaxation dynamics of hard-core boson in 1D

Hardcore bosons in one dimension

\[ \hat{H} = \sum_{i=1}^{L} \left\{ -t \left( \hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + V \hat{n}_i \hat{n}_{i+1} - t' \left( \hat{b}_i^\dagger \hat{b}_{i+2} + \text{H.c.} \right) + V' \hat{n}_i \hat{n}_{i+2} \right\} \]

Relaxation dynamics of hard-core boson in 1D

Hardcore bosons in one dimension

\[
\hat{H} = \sum_{i=1}^{L} \left\{ -t \left( \hat{b}_{i}^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + V \hat{n}_{i} \hat{n}_{i+1} - t' \left( \hat{b}_{i}^\dagger \hat{b}_{i+2} + \text{H.c.} \right) + V' \hat{n}_{i} \hat{n}_{i+2} \right\}
\]


Equilibrium properties

**Nearest-neighbor model**

\[ t' = V' = 0 \]

Integrable (XXZ chain)

Half filling: \[ N_b = \frac{L}{2} \]

S-I Phase transition \( V = 2t \)

Other fillings: \( N_b \neq \frac{L}{2} \)

Superfluid phase for all \( V, t \)

**Next-nearest-neighbor model**

\[ t', V' \neq 0 \]

Nonintegrable

Half filling: \[ N_b = \frac{L}{2} \]

Competing phases

Other fillings: \( N_b \neq \frac{L}{2} \)

Superfluid phase for small \( V', t' \)
Relaxation dynamics of hard-core boson in 1D

Hardcore bosons in one dimension

\[ \hat{H} = \sum_{i=1}^{L} \left\{ -t \left( \hat{b}_{i}^{\dagger} \hat{b}_{i+1} + \text{H.c.} \right) + V \hat{n}_{i} \hat{n}_{i+1} - t' \left( \hat{b}_{i}^{\dagger} \hat{b}_{i+2} + \text{H.c.} \right) + V' \hat{n}_{i} \hat{n}_{i+2} \right\} \]


Nonequilibrium dynamics in 1D

\[ N_{b} = 8 \text{ bosons} \]
\[ N = 24 \text{ lattice sites} \]

Hilbert space: \( H = 735, 471 \)

Largest \( k \)-sector: \( D = 30, 667 \)

Fix \( t' = V' \) and quench

\[ t_{ini} = 0.5, V_{ini} = 2 \]

\[ \rightarrow t_{fin} = 1, V_{fin} = 1 \]

All \( k = 0 \) states are used!
Relaxation dynamics of hard-core boson in 1D

Hardcore bosons in one dimension

\[ \hat{H} = \sum_{i=1}^{L} \left\{ -t \left( \hat{b}^\dagger_i \hat{b}_{i+1} + \text{H.c.} \right) + V \hat{n}_i \hat{n}_{i+1} - t' \left( \hat{b}^\dagger_i \hat{b}_{i+2} + \text{H.c.} \right) + V' \hat{n}_i \hat{n}_{i+2} \right\} \]


Nonequilibrium dynamics in 1D

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All \( k = 0 \) states are used!
Results for $L = 24, N_b = 8$

Relative differences

$$\delta n_k(\tau) = \frac{\sum_k |n(k, \tau) - n_{\text{diag}}(k)|}{\sum_k n_{\text{diag}}(k)}$$

Effective temperature $T = 3.0$

$$E = Z^{-1} \text{Tr} \left\{ \hat{H} \exp(-\hat{H}/k_B T) \right\}$$
Results for $L = 21$, $N_b = 7$

Relative differences

$$\delta n_k(\tau) = \frac{\sum_k |n(k, \tau) - n_{\text{diag}}(k)|}{\sum_k n_{\text{diag}}(k)}$$

Effective temperature $T = 3.0$

$$E = Z^{-1} \text{Tr} \left\{ \hat{H} \exp(-\hat{H}/k_B T) \right\}$$
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4. Summary

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Canonical calculation

\[ A = \text{Tr} \left\{ \hat{A} \hat{\rho} \right\} \]

\[ \hat{\rho} = Z^{-1} \exp \left( -\frac{\hat{H}}{k_B T} \right) \]

\[ Z = \text{Tr} \left\{ \exp \left( -\frac{\hat{H}}{k_B T} \right) \right\} \]

\[ E_0 = \text{Tr} \left\{ \hat{H} \hat{\rho} \right\} \quad T = 3.0J \]
Statistical description after relaxation (nonintegrable)

**Canonical calculation**

\[ A = \text{Tr}\left\{ \hat{A} \hat{\rho} \right\} \]
\[ \hat{\rho} = Z^{-1} \exp\left( -\hat{H}/k_B T \right) \]
\[ Z = \text{Tr}\left\{ \exp\left( -\hat{H}/k_B T \right) \right\} \]
\[ E_0 = \text{Tr}\left\{ \hat{H} \hat{\rho} \right\} \quad T = 3.0J \]

**Microcanonical calculation**

\[ A = \frac{1}{N_{\text{states}}} \sum_{\alpha} \langle \psi_\alpha | \hat{A} | \psi_\alpha \rangle \]

with \( E_0 - \Delta E < E_\alpha < E_0 + \Delta E \)

\( N_{\text{states}} \): # of states in the window
Breakdown of thermalization

\[ L = 24, \ N_b = 8 \]

- **Diff. Diag. vs Microcan.**
  - \( T=2.0, n(k) \)
  - \( T=2.0, N(k) \)
  - \( T=3.0, n(k) \)
  - \( T=3.0, N(k) \)

- **Diff. two initial states**
  - \( T=2.0, n(k) \)
  - \( T=2.0, N(k) \)
  - \( T=3.0, n(k) \)
  - \( T=3.0, N(k) \)
Breakdown of thermalization

$L = 24, N_b = 8$

$L = 21, N_b = 7$
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Paradox?

\[ \sum_{\alpha} |C_\alpha|^2 A_{\alpha\alpha} = \langle A \rangle_{\text{microcan.}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}} \sum_{|E_0 - E_\alpha| < \Delta E} A_{\alpha\alpha} \]

Left hand side: Depends on the initial conditions through \( C_\alpha = \langle \Psi_\alpha | \psi_I \rangle \)
Right hand side: Depends only on the initial energy
Eigenstate thermalization hypothesis

Paradox?

\[ \sum_{\alpha} |C_{\alpha}|^2 A_{\alpha \alpha} = \langle A \rangle_{\text{microcan.}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}} \sum_{|E_0 - E_{\alpha}|<\Delta E} A_{\alpha \alpha} \]

Left hand side: Depends on the initial conditions through \( C_{\alpha} = \langle \Psi_{\alpha} | \psi_I \rangle \)
Right hand side: Depends only on the initial energy

i) For physically relevant initial conditions, \( |C_{\alpha}|^2 \) practically do not fluctuate.

ii) Large (and uncorrelated) fluctuations occur in both \( A_{\alpha \alpha} \) and \( |C_{\alpha}|^2 \). Any physically relevant initial state performs an unbiased sampling of \( A_{\alpha \alpha} \).
Eigenstate thermalization hypothesis

Paradox?

\[ \sum_{\alpha} |C_{\alpha}|^2 A_{\alpha\alpha} = \langle A \rangle_{\text{microcan.}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}} \sum_{|E_0 - E_\alpha| < \Delta E} A_{\alpha\alpha} \]

**Left hand side:** Depends on the initial conditions through \( C_{\alpha} = \langle \Psi_{\alpha} | \psi_I \rangle \)

**Right hand side:** Depends only on the initial energy

---

**State 1**

- \( E_0 = -4.62, T = 3 \)

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**State 2**

- \( E_0 = -4.62, T = 3 \)
Eigenstate thermalization hypothesis

Paradox?

\[ \sum_{\alpha} |C_{\alpha}|^2 A_{\alpha\alpha} = \langle A \rangle_{\text{microcan.}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}} \sum_{|E_0 - E_{\alpha}| < \Delta E} A_{\alpha\alpha} \]

Left hand side: Depends on the initial conditions through \( C_{\alpha} = \langle \Psi_{\alpha} | \psi_I \rangle \)
Right hand side: Depends only on the initial energy

Eigenstate thermalization hypothesis (ETH)
[Deutsch, PRA 43 2046 (1991); Srednicki, PRE 50, 888 (1994); Rigol, Dunjko, and Olshanii, Nature 452, 854 (2008).]

iii) The expectation value \( \langle \Psi_{\alpha} | \hat{A} | \Psi_{\alpha} \rangle \) of a few-body observable \( \hat{A} \) in an eigenstate of the Hamiltonian \( | \Psi_{\alpha} \rangle \), with energy \( E_{\alpha} \), of a large interacting many-body system equals the thermal average of \( \hat{A} \) at the mean energy \( E_{\alpha} \):

\[ \langle \Psi_{\alpha} | \hat{A} | \Psi_{\alpha} \rangle = \langle A \rangle_{\text{microcan.}}(E_{\alpha}) \]
ETH – far away from integrability ($t' = V' = 0.24$)

**Momentum distribution**

Eigenstates $a - d$ are the ones with energies closest to $E_0$.
ETH – far away from integrability \((t' = V' = 0.24)\)

Momentum distribution

Eigenstates \(a - d\) are the ones with energies closest to \(E_0\)

\[
n(k_x = 0) \text{ vs energy}
\]

\[
\rho(E) = P(E) \times \text{dens. stat.}
\]

\[
P(E)_{\text{exact}} \rightarrow |C_\alpha|^2
\]

\[
P(E)_{\text{mic.}} \rightarrow \text{constant}
\]

\[
P(E)_{\text{can.}} \rightarrow \exp\left(-\frac{E}{k_B T}\right)
\]
Breakdown of ETH \(\rightarrow\) integrability \((t' = V' = 0.03)\)

Momentum distribution

Eigenstates \(a - d\) are the ones with energies closest to \(E_0\)

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n(k_x = 0) \text{ vs energy}
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\[
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\]
Relation between thermalization and ETH

Quantifying ETH

$$\Delta_{\text{mic}} O = \sum_\alpha \frac{|O_{\alpha\alpha} - O_{\text{mic}}|}{N_{\text{states}} O_{\text{mic}}}$$

$O_{\alpha\alpha}$: eigenstate expectation values of $\hat{O}$

$O_{\text{mic}}$: microcanonical expectation values of $\hat{O}$

The sum over $\alpha$ contains all states with energies in the window $[E - \Delta E, E + \Delta E]$, and $N_{\text{states}}$ is the number of states in the sum ($\Delta E = 0.1$).

Observables of interest: $n(k = 0)$ and $N(k = \pi)$
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Relative differences

$$\delta n_k = \frac{\sum_k |n(k) - n_{diag}(k)|}{\sum_k n_{diag}(k)}$$

Effective temperature $T = 3.0$

$$E = Z^{-1} \text{Tr} \left\{ \hat{H} \exp(-\hat{H}/k_B T) \right\}$$
Time fluctuations

Are they small because of dephasing?

\[ \langle \hat{A}(t) \rangle - \langle \hat{A}(t) \rangle = \sum_{\alpha', \alpha} C_{\alpha'}^* C_{\alpha} e^{i(E_{\alpha'} - E_{\alpha})t} A_{\alpha', \alpha} \sim \sum_{\alpha', \alpha} e^{i(E_{\alpha'} - E_{\alpha})t} \frac{N_{\text{states}}}{N_{\text{states}}} A_{\alpha', \alpha} \]

\[ \sim \sqrt{N_{\text{states}}^2} \frac{N_{\text{states}}}{N_{\text{states}}} A_{\alpha', \alpha} \sim A_{\alpha', \alpha} \]
Time fluctuations

Are they small because of dephasing?

\[
\langle \hat{A}(t) \rangle - \langle \hat{A}(t) \rangle = \sum_{\alpha', \alpha, \alpha' \neq \alpha} C_{\alpha'}^* C_{\alpha} e^{i(E_{\alpha'} - E_{\alpha})t} A_{\alpha' \alpha} \sim \sum_{\alpha', \alpha, \alpha' \neq \alpha} e^{i(E_{\alpha'} - E_{\alpha})t} \frac{1}{N_{\text{states}}} A_{\alpha' \alpha} \\
\sim \sqrt{\frac{N_{\text{states}}^2}{N_{\text{states}}}} A_{\alpha' \alpha}^{\text{typical}} \sim A_{\alpha' \alpha}^{\text{typical}}
\]

Time average of \( \langle \hat{A} \rangle \)

\[
\langle \hat{A} \rangle = \sum_{\alpha} |C_{\alpha}|^2 A_{\alpha \alpha} \\
\sim \sum_{\alpha} \frac{1}{N_{\text{states}}} A_{\alpha \alpha} \sim A_{\alpha \alpha}^{\text{typical}}
\]
Time fluctuations

Are they small because of dephasing?

\[
\langle \hat{A}(t) \rangle - \langle \hat{A}(t) \rangle = \sum_{\alpha', \alpha} C_{\alpha'}^* C_{\alpha} e^{i(E_{\alpha'} - E_{\alpha})t} A_{\alpha'\alpha} \sim \sum_{\alpha', \alpha} \frac{e^{i(E_{\alpha'} - E_{\alpha})t}}{N_{\text{states}}} A_{\alpha'\alpha}
\]

\[
\sim \frac{\sqrt{N_{\text{states}}^2}}{N_{\text{states}}} A_{\text{typical}}^{\alpha'\alpha} \sim A_{\text{typical}}^{\alpha'\alpha}
\]

Time average of \( \langle \hat{A} \rangle \)

\[
\overline{\langle \hat{A} \rangle} = \sum_{\alpha} |C_{\alpha}|^2 A_{\alpha\alpha}
\]

\[
\sim \sum_{\alpha} \frac{1}{N_{\text{states}}} A_{\alpha\alpha} \sim A_{\text{typical}}^{\alpha\alpha}
\]
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Exact approach to hard-core bosons in 1D lattices

HCB Hamiltonian in an external potential

\[ H = -t \sum_i \left( b_i^\dagger b_{i+1} + h.c. \right) + V_\alpha \sum_i x_\alpha^i n_i \]

Constraints on the bosonic operators

\[ b_i^{\dagger 2} = b_i^2 = 0 \]

Jordan-Wigner transformation

\[ b_i^\dagger = f_i^\dagger \prod_{\beta=1}^{i-1} e^{-i\pi f_\beta^\dagger f_\beta}, \quad b_i = \prod_{\beta=1}^{i-1} e^{i\pi f_\beta^\dagger f_\beta} f_i \]

Non-interacting fermion Hamiltonian

\[ H_F = -t \sum_i \left( f_i^\dagger f_{i+1} + h.c. \right) + V_\alpha \sum_i x_\alpha^i n_i^f \]
Exact approach to hard-core bosons in 1D lattices

One-particle Green’s function

\[ G_{ij} = \langle \Psi_{HCB} | b_i b_j^\dagger | \Psi_{HCB} \rangle = \langle \Psi_F | \prod_{\beta=1}^{i-1} e^{i\pi f_\beta^\dagger f_\beta} f_i f_j^\dagger \prod_{\gamma=1}^{j-1} e^{-i\pi f_\gamma^\dagger f_\gamma} | \Psi_F \rangle \]

\[ \Downarrow \]

Time evolution

\[ | \Psi_F(\tau) \rangle = e^{-iH_F \tau / \hbar} | \Psi_I^F \rangle = \prod_{\delta=1}^{N_b} \sum_{\sigma=1}^{N} P_{\sigma \delta}(\tau) f_\sigma^\dagger | 0 \rangle \]

\[ \Downarrow \]

Exact Green’s function

\[ G_{ij}(\tau) = \det \left[ \left( P^A(\tau) \right)^\dagger \right] \]

Computation time \( \sim N^2 N_b^3 \rightarrow \) study very large systems

3000 lattice sites, 300 particles

Relaxation dynamics in an integrable system

Relaxation dynamics in an integrable system

Statistical description after relaxation

Thermal equilibrium

\[ \hat{\rho} = Z^{-1} \exp \left[ - \left( \hat{H} - \mu \hat{N}_b \right) / k_B T \right] \]

\[ Z = \text{Tr} \left\{ \exp \left[ - \left( \hat{H} - \mu \hat{N}_b \right) / k_B T \right] \right\} \]

\[ E = \text{Tr} \left\{ \hat{H} \hat{\rho} \right\}, \quad N_b = \text{Tr} \left\{ \hat{N}_b \hat{\rho} \right\} \]

MR, PRA 72, 063607 (2005).
**Statistical description after relaxation**

**Thermal equilibrium**

\[
\hat{\rho} = \frac{1}{Z} \exp \left[ - \frac{\left( \hat{H} - \mu \hat{N}_b \right)}{k_B T} \right]
\]

\[
Z = \text{Tr} \left\{ \exp \left[ - \frac{\left( \hat{H} - \mu \hat{N}_b \right)}{k_B T} \right] \right\}
\]

\[
E = \text{Tr} \left\{ \hat{H} \hat{\rho} \right\}, \quad N_b = \text{Tr} \left\{ \hat{N}_b \hat{\rho} \right\}
\]

MR, PRA 72, 063607 (2005).

**Evolution of** \( n_{k=0} \)

**\( n_k \) after relaxation**

Evolution of \( n_{k=0} \)

\( n_k = 0 \) to 1000 2000 3000 4000

\( \tau \)

\( n_{k=0} \)

0 0.5 1 1.5

\( \text{Time evolution} \)

\( \text{Thermal} \)

\( n_k \) after relaxation

\( n_k = 0 \) to \( \pi/2 \) \( \pi/2 \) \( \pi \)

\( n_k \)

0 0.25 0.5

\( \text{After relax.} (N_b=30) \)

\( \text{After relax.} (N_b=15) \)

\( \text{Thermal} (N_b=30) \)

\( \text{Thermal} (N_b=15) \)
Statistical description after relaxation

**Thermal equilibrium**

\[ \hat{\rho} = Z^{-1} \exp \left[ - \left( \hat{H} - \mu \hat{N}_b \right) / k_B T \right] \]

\[ Z = \text{Tr} \left\{ \exp \left[ - \left( \hat{H} - \mu \hat{N}_b \right) / k_B T \right] \right\} \]

\[ E = \text{Tr} \left\{ \hat{H} \hat{\rho} \right\} , \quad N_b = \text{Tr} \left\{ \hat{N}_b \hat{\rho} \right\} \]

MR, PRA 72, 063607 (2005).

**Constrained equilibrium**

\[ \hat{\rho}_c = Z_c^{-1} \exp \left[ - \sum_m \lambda_m \hat{I}_m \right] \]

\[ Z_c = \text{Tr} \left\{ \exp \left[ - \sum_m \lambda_m \hat{I}_m \right] \right\} \]

\[ \langle \hat{I}_m \rangle_{\tau=0} = \text{Tr} \left\{ \hat{I}_m \hat{\rho}_c \right\} \]
Statistical description after relaxation

Thermal equilibrium

\[ \hat{\rho} = Z^{-1} \exp \left[ - \left( \hat{H} - \mu \hat{N}_b \right) / k_B T \right] \]

\[ Z = \text{Tr} \left\{ \exp \left[ - \left( \hat{H} - \mu \hat{N}_b \right) / k_B T \right] \right\} \]

\[ E = \text{Tr} \left\{ \hat{H} \hat{\rho} \right\}, \quad N_b = \text{Tr} \left\{ \hat{N}_b \hat{\rho} \right\} \]

MR, PRA 72, 063607 (2005).

Constrained equilibrium

\[ \hat{\rho}_c = Z_c^{-1} \exp \left[ - \sum_m \lambda_m \hat{I}_m \right] \]

\[ Z_c = \text{Tr} \left\{ \exp \left[ - \sum_m \lambda_m \hat{I}_m \right] \right\} \]

\[ \langle \hat{I}_m \rangle_{\tau=0} = \text{Tr} \left\{ \hat{I}_m \hat{\rho}_c \right\} \]
Relaxation dynamics in an integrable system

Density profile

Momentum profile

\( \tau = 2000t \)

Constrained

Marcos Rigol (Georgetown University)

Breakdown of thermalization in 1D

September 17, 2009 32 / 35
Integrals of motion
(underlying noninteracting fermions)

\[ \hat{H}_F \hat{\gamma}_m^{\dagger} |0\rangle = E_m \hat{\gamma}_m^{\dagger} |0\rangle \]

\[ \{ \hat{I}_m \} = \{ \hat{\gamma}_m^{\dagger} \hat{\gamma}_m \} \]

Lagrange multipliers

\[ \lambda_m = \ln \left[ \frac{1 - \langle \hat{I}_m \rangle_{\tau=0}}{\langle \hat{I}_m \rangle_{\tau=0}} \right] \]
Statistical description after relaxation

Integrals of motion
(underlying noninteracting fermions)

\[ \hat{H}_F \hat{\gamma}_m^{f\dagger} |0\rangle = E_m \hat{\gamma}_m^{f\dagger} |0\rangle \]
\[ \{ \hat{I}_m \} = \{ \hat{\gamma}_m^{f\dagger} \hat{\gamma}_m^f \} \]

Lagrange multipliers

\[ \lambda_m = \ln \left[ \frac{1 - \langle \hat{I}_m \rangle_{\tau=0}}{\langle \hat{I}_m \rangle_{\tau=0}} \right] \]

Other examples in:

- M. Cramer et al., PRL 100, 030602 (2008)
There is thermalization far away from integrability

★ Finite size effects
Summary

- There is thermalization far away from integrability
  - Finite size effects
- Eigenstate thermalization hypothesis
  - \( \langle \psi_\alpha | \hat{A} | \psi_\alpha \rangle = \langle A \rangle_{\text{microcan.}} (E_\alpha) \)
Summary

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- Integrable systems are different (Generalized Gibbs ensemble)
Poincaré recurrences?