Chapter 4

Above threshold ionization in atomic systems

4.1 Outline

4.1.1 Historical overview

This chapter intends to give a brief description of what is called Above Threshold Ionization (ATI) in connection to other aspects of the interaction between atoms and laser radiation. Excellent reviews on this subject, including other phenomena in intense laser fields, were given by Protopapas et al. [47], Joachain et al. [48] and, more recently, by Becker et al. [49].

We begin by mentioning the multiphoton (single) ionization (MPI), whereby an atom or an ion absorbs several photons from the laser field causing the weakly bound electron to ionize. In 1965, Voronov and Delone [50] observed this process using a ruby laser to induce seven-photon ionization of xenon.

In early experiments MPI was accurately modelled by the lowest-order perturbation theory (LOPT) (Fabre et al. [51], Petite et al. [52]), the n-photon ionization rate being given by

\[ \Gamma_n = \sigma_n I^n \]
where \( n \) is the minimum number of photons needed for ionization, \( \sigma_n \) is the generalized cross section and \( I \) is the intensity of the incident light. Difficulties arise with the perturbative approach as the intensity of the incident light increases and therefore, due to strong coupling with the laser field, the atomic states can no longer be considered as unperturbed.

An important step in our understanding of MPI came along with improved experimental techniques, giving the possibility of detecting energy-resolved photoelectrons. It allowed Agostini et al. [53] to discover that the ejected electron could absorb photons in excess of the minimum required for ionization to occur. This became known as ‘above threshold ionization’ (ATI), and has been largely studied over the recent years.

A typical ATI photo-electron energy spectrum consists of several peaks, separated by the photon energy \( \hbar \omega \). As the intensity \( I \) increases, peaks at higher energies appear, whose intensity dependence does not follow the power law according to lowest order perturbation theory (LOPT).

Another feature of ATI spectra in low frequency fields, is that as the intensity increases, the low-energy peaks reduce in magnitude, as the energies of the atomic states are Stark-shifted in the presence of the laser field. In the length gauge picture, the AC Stark shifts of the lowest bound states are smaller than those for the Rydberg and continuum states. Because the energies of the continuum states are shifted upwards relative to the lower bound states, there is a corresponding increase in the intensity-dependent ionization potential of the atom. This increase is essentially given by the electron ponderomotive energy \( U_p \), which is the cycle-averaged kinetic energy of a quivering electron in a laser field of frequency \( \omega \). For non-relativistic velocities,

\[
U_p = \frac{e^2 E_0^2}{4m\omega^2} \tag{4.1}
\]

where \( e \) and \( m \) are the electron’s charge and mass, and \( E_0 \) is the electric field amplitude.

Novel structures in the ATI spectra have been identified in recent experiments.
Usually the angular distributions of electrons in the ATI spectrum produced by very intense laser fields are aligned along the axis of polarization of the applied laser field. Two groups (Yang et al. [54], Feldmann [55]) have shown that for noble gases a few high energy peaks may be highly structured, and in some cases feature rings 45° off the polarization axis. Theoretical investigations have shown that these rings may arise from the rescattering of the electron wavepacket from the parent ion (Kulander and Schafer [14], Paulus et al. [56], Lewenstein et al. [57]). The experiment of Paulus et al. [58] showed the existence of the rescattering plateau in the ATI spectrum and one- and three-dimensional theoretical simulations demonstrated that this effect is of single-atom nature. The connection with the high energy side lobes was given by Paulus, who found that the side lobes were restricted to the regions where the plateau begins.

### 4.1.2 Theoretical methods

The single-active-electron approximation (SAE) is widely used in atomic physics. It consists of modelling the atom in the laser field by a single electron that interacts with the laser field and is bound by an effective potential. This potential is optimized so as to reproduce the ground state and singly excited states.

In single ionization in strong fields, up to now, no effect has been identified that would reveal electron-electron correlation so that the ATI spectra are, in essence, not influenced by multi-electron correlation effects. This can be seen in a comparison made by Nandor et al. [2], where the experimental result is compared to the prediction of the three-dimensional time-dependent Schrödinger equation (TDSE). Very good agreement is found (see Fig. 4.1).

Although numerical solution of the one-particle TDSE in one dimension proved to be fruitful in early days to understand ATI, for further investigation the three-dimensional solution is needed. This is computationally very demand-
Figure 4.1: (a) Measured and (b) calculated photoelectron spectrum in argon for 800 nm, 120 fs pulses at the intensities given in TW/cm² in the figure ($10U_p = 39$ eV). From Nandor et al. [2].

The progress achieved in recent years in the intuitive understanding of the ATI is mainly due to the Simpleman’s model and its refinements, a theory that analyzes the behavior of classical electron trajectories in an oscillating electric field of a laser. In its initial form [59], the Simpleman’s model explains the photoelectron spectra of the so-called ‘direct’ electrons (electrons that don’t rescatter off the atomic core) and predicts an upper limit of $2U_p$ for their kinetic energy in the case of short laser pulses. The validity of the model has been confirmed in microwaves experiments [60].

Along with new computational techniques being devised to complete the difficult numerical task, other analytic approaches apart from the Simpleman’s model have been proposed. The starting point of the Keldysh-type theories, which gained a lot of attention, is the existence of an analytical solution for a free electron in a plane-wave laser field, the Volkov solution [61]. As the laser
field is of high intensity, the binding potential can be treated as a perturbation. The ground state is propagated with the operator of the laser field and projected onto the Volkov states. Further improvement of these theories include an additional interaction with the potential in order to describe the rescattering of the electron. Later, Lewenstein et al. decomposed the Keldysh rate into the contributions of the relevant trajectories in the spirit of Feynman’s path integrals [57]. This showed that the Keldysh-type theories are related to the classical Simpleman’s model picture, and indicated also that the mysterious dips in the ATI spectra are due to interference between electron’s trajectories. These methods are particularly useful for theoretical investigations in the case of stronger laser fields, longer pulses and low frequency, where a direct TDSE solution proves to be increasingly prohibitive.

4.2 Direct ionization in a stationary field

A stationary laser pulse is usually considered as a plane wave and modelled by the vector potential $\mathbf{A}(t) = A_0 \cos(\omega t) \mathbf{\hat{e}}$, directed along the axis of polarization $\mathbf{\hat{e}}$. The magnitude of the electric field is $E_0 = A_0 / \omega$, where $\omega$ is the radiation frequency. The pulse has an infinite number of cycles with duration $T_p = 2\pi/\omega$. Because of this periodicity, we can define a rate of ionization as the ionization occurring per optical cycle. The cycle-averaged kinetic energy of an electron in such a field reads $U_p = A_0^2 / 4$, which is the equivalent of the expression (4.1) in atomic units.

4.2.1 The quantum-mechanical description

The transition amplitude for the direct electrons – electrons that leave the vicinity of the atom right after they have tunneled into the continuum – is the well known Keldysh-Faisal-Reiss (KFR) amplitude [62–64]. This approximation neglects the binding potential in the propagation of the electron in
the continuum, and the laser field when the electron is bound. The transition amplitude reads [65]:

\[ M_p^{(0)} = -i \int_{-\infty}^{\infty} dt \langle \psi_p^{VI}(t) | V | \psi_0(t) \rangle, \quad (4.2) \]

where \( V \) denotes the atomic binding potential and the initial state is the usual hydrogenic ground state with atomic number \( Z \) and ionization energy \( I_p = Z^2 / 2 \):

\[ \langle r | \psi_0(t) \rangle = \frac{Z^3}{2 \sqrt{\pi}} e^{-Zr} e^{-iI_p t}. \]

The final state is approximated with the Volkov state [see Appendix (A)] describing a charged particle with asymptotic momentum \( p \) in the presence of a field with vector potential \( A(t) \). In the length gauge, we have:

\[ \langle r | \psi_p^{VI}(t) \rangle = \frac{e^{i r \cdot [p + A(t)]}}{(2\pi)^{3/2}} \exp \left\{ i \int_{t}^{\infty} d\tau \frac{[p + A(\tau)]^2}{2} \right\} \cdot \quad (4.3) \]

After some manipulations, which are detailed in Subsection 5.3.1, the integral in (4.2) can be written in the form:

\[ M_p^{(0)} = i \frac{Z^{5/2}}{\pi \sqrt{2}} \int_{0}^{T_p} dt \frac{\exp[iS(t)]}{S'(t)}, \quad (4.4) \]

where \( T_p = n_p 2\pi / \omega \) is the duration of \( n_p \) optical cycles of the finite laser pulse. The time integration domain is changed compared to (4.2), which describes in fact a stationary field. For the stationary field case, the integral can be replaced by an integral over just one cycle of the field, with the additional condition of energy conservation; the conservation of energy for the ejected electron reads \( E_p \equiv p_n^2 / 2 = n\omega - (I_p + U_p) \), where \( n \) is the number of photons absorbed. It can be attributed to the interference of the contributions from different periods, i.e., the quantum interference is destructive, unless the energy is conserved.

We introduced, in Eq. (4.4), the quantity

\[ S(t) = \frac{1}{2} \int_{0}^{t} d\tau [p + A(\tau)]^2 + I_p t, \quad (4.5) \]

which is the modified classical action of a free electron with conserved canonical momentum \( p \) in the field. The dependence of the action \( S \) on the \( p \) variable has been dropped here for clarity.
A valuable physical insight can be drawn from Eq. (4.4) when applying the saddle point method. To find the saddles, one has to solve the equation \( dS(p, t)/dt = 0 \), which can be done analytically for the stationary field. There are only two solutions in each laser cycle, with positive imaginary parts. Their trajectories in the complex plane are shown in Fig. 4.2.

![Saddle point trajectories](image)

**Figure 4.2**: The saddle point trajectories in the complex plane for a He\(^+\) ion irradiated by a stationary pulse with intensity \( I = 10^{16} \text{ W cm}^{-2} \) and wavelength of 800 nm. The electron is emitted along the polarization axis. The dashed curve is the profile of the vector potential during one optical cycle. The circles represent the positions of the two saddles for \( p=0 \) and the end points of the trajectories correspond to \( p = A_0 \).

For low kinetic energy, the two saddles are close to the maximum of the electric field (or, alternatively, to the zeros of the vector potential) and with increasing kinetic energy, their imaginary part increases, causing the ionization amplitude to decrease exponentially. (We discuss later this aspect.)

The advantage of the saddle point method lays in the fact that the complicated integral that represents the amplitude of ionization can be reduced to a sum over just two terms. This simplification makes it easier to analyze ionization within our model, leading to a semiclassical model, which we describe in
Subsection (4.2.2).

While for the infinitely long monochromatic pulse, the ionization amplitude is invariant with respect to the transformation $p \rightarrow -p$, the backward-forward symmetry in no longer true for a finite pulse. This may help in determination of the absolute carrier phase (Dietrich et al. [66]).

### 4.2.2 The classical model of ionization

For a further detailed technical discussion of the amplitude integral, see Eqs. (E.1) and (E.14). The latter, which is derived by using the simplified version of the saddle point method, we reproduce below, with the appropriate pre-factors:

$$M_p^{(0)} = -Z^{5/2} \sum_{n_0=1,2} \frac{\pi \exp[iS(t_0)]}{(2I_p + p_\perp^2)^{1/2}|E(t_0)|} \exp \left[ -\frac{1}{3} \frac{(2I_p + p_\perp^2)^{3/2}}{|E(t_0)|} \right]. \quad (4.6)$$

Here, $p_\perp$ is the component of the electron’s asymptotic momentum $p$, perpendicular to the polarization axis. This expression provides an intuitive way to describe semiclassically the phenomenon of ionization in the Simpleman’s

![Figure 4.3: The graphical solution for the ‘birth’ times \{t_{01}, t_{02}\} for a stationary field, satisfying $p_\parallel/A_0 + \cos(\omega t) = 0$ with $p_\parallel = -0.5 A_0$.](image)
model for direct ionization, in which the electron is ‘born’ in the continuum at real times \( t_0 \) given by \( p_\parallel + A(t_0) = 0 \). A graphical solution is presented in Fig. 4.3, for \( p_\parallel = -0.5A_0 \).

The condition states that the electron is ‘born’ with zero kinetic energy along the polarization axis and also sets a classical cutoff value for this energy: a solution \( t_0 \) exists only if \( p_\parallel \leq A_0 \rightarrow E_{p_\parallel} \equiv \frac{p_\parallel^2}{2} \leq \frac{A^2_0}{2} \).

After ionization, the second step of the classical model is the evolution of the electron in the laser field. As the electron’s oscillation amplitude is much larger than the atomic distances, the influence of the binding potential can be ignored. For a given vector potential \( A(t) \) and a canonical momentum \( p \), which is conserved, the electron’s velocity reads:

\[
\mathbf{v}(t) = \mathbf{p} + A(t).
\]  

(4.7)

The velocity consists of a constant term \( \mathbf{p} = -A(t_0) \), which is the drift momentum measured at the detector, and an oscillating term. The kinetic energy of the electron, averaged over a laser field period \( T_p \), is:

\[
E_p = \frac{\langle \mathbf{v}^2(t) \rangle_{T_p}}{2} = \frac{p^2}{2} + \frac{\langle A^2(t) \rangle_{T_p}}{2} = E_{\text{drift}} + U_p.
\]  

(4.8)

The ponderomotive energy is often employed to characterize the laser intensity. Now, we can re-write the classical cutoff for the energy of direct electrons as \( E_p \leq 2U_p \). In the quantal model, this bound is softened. However, it is useful as a guideline in the analysis of experimental spectra.

Of the two electrons, born at times \( t_{01} \) or \( t_{02} \) as depicted in Fig. 4.3, one keeps moving directly away from the atom, never crossing the atomic core, while the other starts moving in the opposite direction, turns around at a later time crossing the atomic core, to acquire in the end the same drift momentum as the first electron. The two different types of trajectories could relate to the ‘direct’ and ‘indirect’ wavepackets of de Bohan et al. in Ref. [4]. The emission of the electron at each of the two ionization times \( t_0 \) is weighted with a probability
containing the real exponential factor
\[
\exp \left[ -\frac{1}{3} \frac{(2I_p + p_{\perp}^2)^{3/2}}{|E(t_0)|} \right].
\]

The factor decreases with \( p_{\perp} \), so that the probability of ejection with high velocities in a direction perpendicular to the polarization direction is exponentially small. This term was derived by Delone and Krainov [67].

We note that the terms in the sum (4.6) are in the form of a real amplitude (containing the exponentially decreasing factor) and a complex phase factor \( \exp[\pm i S(t_0)] \). Because of the symmetry of the pulse, the real factors are the same for the two birth times and they can be factorized; this way, the phase terms add and become responsible for interference in the energy spectrum, creating a beat pattern. Because of its sensitive dependence on the laser intensity, which is not very well controlled in an experiment, the beat pattern is usually washed out.

Apart from the exponentially decreasing term, the other real pre-factors can differ from one expression for the ionization amplitude to another (depending on whether we use the length gauge or velocity gauge). What is common to all these formulations is the exponentially decreasing term and the phase factor.

4.2.3 Interferences of direct electrons

For a stationary field, during one optical cycle, there are two saddle points corresponding to a drift momentum \( \mathbf{p} \). For small momenta, the two satisfy the approximate relation:
\[
\cos(\omega t_{s1,2}) = \pm i \gamma,
\]
where \( \gamma \equiv \sqrt{2I_p/A_0} \) is the Keldysh parameter. Both complex saddles have the real part close to the moments in time when the electric field reaches the maximum value, at \( T_p/4 \) and \( 3T_p/4 \) (here \( T_p \) is the duration of one laser period). The classical trajectory for an electron starting at the origin with zero initial kinetic energy is \( \mathbf{r}(t) = (A_0/\omega) \sin(\omega t) \). Hence, we can find the electron
positions at the two instants of detachment $t_{s1,2}$. Although the detachment times are complex, the positions are real:

$$r_{1,2} \equiv r(t_{s1,2}) = \pm \frac{A_0}{\omega} \sqrt{1 + \gamma^2}. \quad (4.10)$$

From here, as discussed by Gribakin and Kuchiev [68], appears an intuitive two-slit interference picture for the ionization spectrum at low energies. According to this picture, the angular distribution of photoelectrons, emitted at an angle $\theta$ with respect to the polarization direction, is determined by the interference of two plane waves of wavevector $k = \mathbf{p}$ emitted with a time delay $T_p/2$ from two sources separated by the distance $r_1 - r_2$. These points are located at the opposite sides of the atomic particle and emit the waves at an angle $\theta$ with respect to the polarization direction (see Fig. 4.4).

\[\text{Figure 4.4: Schematic picture of the origin of interference in electron emission from two laser-induced sources. The indices (1) and (2) correspond to the two positions as in Eq. (4.10).}\]

The phase difference is obtained by multiplying the distance between the two points by the projection of the electron momentum on the direction of the field $p \cos \theta$. As a result, one obtains for the interference term $2 \frac{A_0}{\omega} p \cos \theta \sqrt{1 + \gamma^2}$. It is remarkable that this term is identical to the one obtained from Eq. (4.6) when doing the explicit calculation [see Eq. (D.9)].
4.3 Rescattering

So far, we have presented calculations for the ‘direct’ electrons, which after the first step of ionization leave the laser focus without any additional interaction with the atomic core. In this section, we will consider the consequences of such an encounter.

4.3.1 The classical theory

The classical model becomes richer if rescattering effects are taken into account. To find the electron trajectory, we integrate its velocity [Eq. (4.7)]:

$$r(t) = \int_{t_0}^{t} A(\tau) d\tau - A(t_0)(t - t_0). \quad (4.11)$$

The condition for the electron to return at a later time $t_1 > t_0$ to the atomic core is given by

$$r(t_1) \equiv \int_{t_0}^{t_1} A(\tau) d\tau - A(t_0)(t_1 - t_0) = 0. \quad (4.12)$$

For linear polarization, it simply states that the electron moves along the polarization axis towards the origin. When it returns, one of the following can happen (Corkum, 1993 [15]):

- The electron may recombine with the core, emitting its energy plus the ionization energy as one photon. This process is responsible for the plateau of high-order harmonic generation.
- The electron may scatter inelastically off the core and dislodge a second electron. This process is now believed to be the dominant contribution to the non-sequential double ionization.
- The electron may scatter elastically, so that it can acquire drift energies much bigger than otherwise.
The kinetic energy of the electron at the time of return is

\[ E_{\text{ret}} = \frac{1}{2} [A(t_1) - A(t_0)]^2. \]  

(4.13)

Maximizing this energy together with the condition (4.12), one gets for a stationary field \( E_{\text{ret, max}} = 3.17U_p \).

With the mechanism of elastic rescattering, where the energy is conserved, but the velocity can change direction, the drift energy increases. The maximal energy is obtained when the electron backscatters by 180°. A rescattering at different angle gives less gain in energy, as part of the maximal energy of the returning electron goes into motion transverse to the polarization axis.

Let us assume that at the time of collision with the core the electron has the speed \( v(t_1 - 0) = A(t_1) - A(t_0) \) along the polarization direction and after \( v(t_1 + 0) = -[A(t_1) - A(t_0)] \). Then, from (4.7) we obtain the drift energy upon backscattering:

\[ E_{bs} = \frac{1}{2} [2A(t_1) - A(t_0)]^2 \]  

(4.14)

Maximizing (4.14) under the condition (4.12) yields \( E_{bs, max} = 10.007U_p \). It is important to note that for a given final energy smaller than the cutoff energy of 10\( U_p \), there are two return times \( t_1 \) within one optical cycle. The two trajectories can be distinguished by the flight time of the electron.

### 4.3.2 The quantum-mechanical description

The generalized transition amplitude, which includes one act of rescattering, is given by [69]:

\[ M_{p} = - \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt' \langle \psi_p V(t)|VU V|\psi_0(t') \rangle \]  

(4.15)

where \( V \) denotes the atomic binding potential and \( U V V(t, t') \) is the Volkov time-evolution operator, describing the evolution of an electron in the presence of the laser field only. In the limit \( t \rightarrow t' \), the rescattering amplitude (4.15) goes into (4.2). Hence, they must not be added. Close versions of the rescattering amplitude have been used by several authors [70–74].
Inserting the expansion of the Volkov propagator in terms of Volkov states,

\[ U^{\text{Vl}}(t, t') = \int d^3k|\psi_k^{\text{Vl}}(t)\rangle\langle \psi_k^{\text{Vl}}(t')| \] (4.16)

into Eq. (4.15), the rescattering transition amplitude can be rewritten as

\[ M_p = -\int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt' \int d^3k e^{iS_p(t, t', k)} V_{pk} V_{k0}, \] (4.17)

with the corresponding action given by

\[ S_p(t, t', k) = -\frac{1}{2} \int_t^{t'} d\tau [p + A(\tau)]^2 \]
\[ -\frac{1}{2} \int_t^{t'} d\tau [k + A(\tau)]^2 + I_p t'. \] (4.18)

The expressions are useful if the form factors

\[ V_{pk} = \langle p + A(t)|V|k + A(t)\rangle \]
\[ = \frac{1}{(2\pi)^3} \int d^3r \exp[-i(p - k) \cdot r] V(r) \] (4.19)

and

\[ V_{k0} = \langle p + A(t')|V|0\rangle \]
\[ = \frac{1}{(2\pi)^3} \int d^3r \exp[-i(k + A(t')) \cdot r] V(r) \psi_0(r) \] (4.20)

can be calculated in analytical form. This is the case for the Yukawa potential

\[ V(r) = -Z \exp(-\alpha r)/r, \]

when we have:

\[ V_{pk} = -\frac{Z^2}{2\pi^2 (p - k)^2 + \alpha^2} \] (4.21)

and

\[ V_{k0} = -\sqrt{2} \frac{Z^{5/2}}{\pi (Z + \alpha)^2 + |k + A(t')|^2} \]
\[ = -\sqrt{2} \frac{Z^{5/2}}{\pi (Z + \alpha)^2 - 2I_p}. \] (4.22)

The Coulomb form factors can be retrieved from Eqs. (4.21) and (4.22) in the limit \( \alpha \to 0 \). Since for this case \( I_p = Z^2/2 \), this leads to the well-known
The interpretation of the amplitude (4.17) can now be given in terms of the form factors. Firstly, the electron tunnels out from the ground state to an intermediate scattering state with free momentum \( k \) [the corresponding matrix element is \( V_{k0} \) in (4.20)]. Once in the continuum, the electron scatters under the influence of the bounding potential from the scattering state with momentum \( k \) to the final state with asymptotic momentum \( p \) [the matrix element is \( V_{pk} \) in (4.19)]. The resulting amplitude is summed over all the possible intermediate momenta \( k \) and integrated along the pulse duration.

Equation (4.17) simplifies by restricting the integration over the rescattering time \( t \) to only one laser cycle together with imposing the conservation of energy condition. The latter is a consequence of the interference of contributions to the ionization amplitude coming from different laser cycles, which is characteristic to the stationary field.

### 4.3.3 Saddle point method. Quantum orbits

For the rescattering amplitude, the saddle point equations for the integration variables \( t, t' \) and \( k \) in Eq. (4.17) are:

\[
\begin{align*}
[k + A(t')]^2 &= -2I_p \quad \text{(4.23a)} \\
[p + A(t)]^2 &= [k + A(t)]^2 \quad \text{(4.23b)} \\
\int_{t'}^t d\tau [k + A(\tau)] &= 0. \quad \text{(4.23c)}
\end{align*}
\]

Their solution determine the ionization time \( t' \), the rescattering time \( t \), and the drift momentum \( k \) of the electronic orbit between those two times, such that in the end the electron acquires the asymptotic momentum \( p \). Equations (4.23a) and (4.23b) are related to the energy conservation at the ionization time and rescattering time, respectively, and equation (4.23c) determines the intermediate electron momentum.
If we let $I_p = 0$ in Eqs. (4.23), all the variables are real and we obtain the same equations as in the Simpleman’s model [75]. One very important feature of the solution for $(t', t)$ is that they come in pairs. If we denote the ‘travel time’ by $\tau \equiv t - t'$, we find that for a given asymptotic momentum $p$ and for the $n$th travel-time interval $nT_p/2 \leq \tau \leq (n + 1)T_p/2$ ($n = 1, 2, \ldots$), there are two solutions having slightly different travel times. A solution $(t', t, k)$ describes an electron that starts from the origin of the binding potential at time $t'$ with drift momentum $k$, returns to the atomic core due to the laser field at time $t$, rescatters and subsequently moves away in the laser field with drift momentum $p$. For increasing electron energy, the two solutions in a pair approach each other to finally become identical. The energy when this happens is the cutoff energy of the pair and it represents the maximum kinetic energy an electron starting with zero velocity can gain classically. (For a more detailed discussion, see [76].)

Numerically solving the system of non-linear equations (4.23) is not an easy task as the numerical routines available require a good initial guess which then they refine to the exact solution. To find this guess, one needs to systematically search the region of the plane $(t', t)$ we are interested in. If we allow for $I_p$ to be non-zero, the times become complex and we have to consider two extra dimensions, the imaginary parts of both times. This makes our task even more difficult.

We have been able to find a simple way to identify the real solutions for a given electron energy $p$. We look for the return times occurring in the same cycle $\omega t \in [0, 2\pi]$ and for ionization times in all the precedent cycles $\omega t' \in [-(n + 1)2\pi, -n2\pi]$, $n = 0, 1, \ldots$. The solutions can be visualized graphically by using the ImplicitPlot command available within Mathematica [77]. In Fig. 4.5 we plot the two implicit curves given by Eqs. (4.23a) and (4.23b) with $I_p = 0$ and for two increasing values of the drift momentum $p$; the increasing of the momentum makes the solutions to approach each other, until they eventually coalesce at the cutoff.
Figure 4.5: The vertical curves satisfy (4.23a) with $I_p = 0$ and the horizontal ones (4.23b) ($k \neq p$). Therefore, the intersection points give the solutions for the trajectories corresponding to final drift momentum $p = 0$ (left panel) and $p^2/2 = 5.8U_p$ (right panel, with $p$ in the polarization direction).

It is obvious from Eq. (4.23a) that as long as $I_p \neq 0$, the times $t, t'$ and the momentun $k$ are complex. Physically, the fact that $t'$ is complex means that the ionization takes place via a tunneling process. Regardless of the electron energy, the orbits are now complex. These are often called quantum orbits or quantum trajectories. Unlike their real counterparts, after the cutoff they do not disappear, but their real parts remain essentially frozen and the imaginary parts quickly increase. Two typical pairs of quantum orbits are displayed in Fig. 4.6.

Both ionization and rescattering times depend on the ionization energy and the photo-electron momentum $p$, but not on the shape of the potential $V(r)$, which enters the transition amplitude only by the form factors (4.21) and (4.22). It is to be expected that for high energy electrons, their spectrum would not be much influenced by the binding potential. This was confirmed by the numerical experiments within the SFA.

Within the saddle-point approximation, the amplitude (4.17) can be approxi-
Figure 4.6: Saddle points for scattering along the polarization axis for a Keldysh parameter $\gamma = 1.11$ and $p = 0$. The labels refer to the fact that the pairs (1,2) and (9,10) are ordered as a function of the increasing travel time $t - t'$. The circles correspond to $p = 0$ and the trajectories end at $p^2 / 2 = 10U_p$.

The saddle point method allows one to calculate the rescattering amplitude, estimated by:

$$M_p \approx \sum_s A_s \exp(iS_s)$$  \hfill (4.24a)

$$S_s = S_p(t_s, t'_s)$$  \hfill (4.24b)

$$A_s = (2\pi i)^{5/2} V^p_{k(t_s, t'_s)} V_{k(t_s, t'_s)0} \sqrt{\det S''_p(t, t')},$$  \hfill (4.24c)

where the index $s$ runs over the relevant saddle points and $\det S''_p(t, t')|_s$ is the five-dimensional matrix of the second derivative of the action $A(t)$ evaluated at the solutions of the saddle point equations (4.23). To avoid including the contribution of the direct electrons, we choose to sum only those pairs of solutions for which the travel time is bigger than half of the laser period. This way, we select only the rescattering trajectories.
avoiding the five dimensional integral present in its definition, which is a consider- 
able simplification. This comes at the expense of solving a system of non-linear, complex equations and dealing with the so-called Stokes transition 
phenomenon encountered when applying the saddle point method. The Stokes 
transition is described in the following.

4.3.4 The saddle point uniform approximation

Upon approach to the classical cutoff, the two solutions that make up a pair 
come very close to each other. As an example, this is illustrated in Fig. 4.6. On 
the other hand, the saddle point approximation (4.23) treats different saddles 
as independent. As mentioned in [65] and the references therein, this leads to 
a qualitative breakdown of the standard saddle point approximation near the 
cutoff of any pair of solutions, mainly for two reasons: (i) This approxima-
tion can overestimate the saddle contribution to the transition amplitude by 
several orders of magnitude (and can actually diverge if both saddle coalesce).
(ii) The usual procedure to avoid the problems described in (i) is to drop, 
after the cutoff, the spurious saddle (the saddle whose contribution increases 
exponentially after the cutoff – this is the Stokes phenomenon) by requiring 
a minimal discontinuity of the transition amplitude. Still, a discontinuity re-
mains visible in the spectrum and it can still affect quantitatively the results 
in an undesirable manner.

The remedy offered by the asymptotic expansion theory comes in the form of 
the uniform approximation. To improve the expansion of the action function 
in the neighborhood of the two saddles in the pair, one needs to include higher 
orders in the coordinate dependence and to take the resulting integrals as the 
collective contribution of both saddles. The final expression can be cast in 
a simple form, which uses the same information as the simple saddle point 
method and this has been done in [65]. The uniform approximation for the 
case of two close saddles $i$ and $j$ using the Airy function and its derivative
reads:

\[ M_{i+j} = (6\pi S_-)^{1/2} \exp(iS_+ + i\pi/4) \times \left[ \frac{A_-}{\sqrt{z}} \text{Ai}(-z) + i \frac{A_+}{z} \text{Ai}'(-z) \right], \tag{4.25} \]

where \( z = (3S_-/2)^{2/3} \) and the other quantities are:

\[ 2S_{\pm} = S_i \pm S_j \]
\[ 2A_{\pm} = A_i \pm iA_j. \tag{4.26} \]

The subscript \( i \) denotes the first saddle in the ordered pair, whose contribution does not increase after the Stokes transition. The uniform contribution given in (4.25) appears as given in [78]. Another, equivalent version, using the Bessel functions is given in [65]. It can be shown that for \( |S_-| \) large (i.e., the saddle points can be treated as independent), (4.25) reduces to (4.24). Also, due to the asymptotic properties of the Airy function for large arguments, the Stokes transition at the classical cutoff is automatically built into the uniform approximation. Notably, the uniform approximation is of the same simplicity as the usual saddle point formula.

By using the uniform approximation, we calculate here the rescattering amplitude for the case of a zero-range potential (see [65], Section VI). We choose the parameters such that we have \( U_p/\omega = 3.58, \omega = 0.05695 \) (800 nm) and a ground-state energy of \( E_0 = -0.5 \) a.u. The calculation is done for emission along the polarization axis.

The output of the calculation is shown in Fig. 4.7, where we can compare the relative magnitudes of the direct ionization and rescattering term with respect to the full amplitude; the difference can be as much as four orders of magnitude. Also, it is important to note that there is an energy interval (between \( 5U_p \) and \( 6U_p \)) where the direct contribution is of the same order of magnitude as the rescattering, so that interference effects can occur.

One other aspect is the cutoff region in the rescattering spectrum, which is around the value of \( 10U_p \) in electron drift energy, after which the magnitude
decreases exponentially. This comes as a confirmation of the intuitive results obtained in the classical model of the rescattering discussed in Subsection 4.3.1.

Had we used the usual saddle point method, due to the Stokes transition not being taken into account properly, cusps would have appeared in the spectrum, with up to an order of height in magnitude, thus significantly altering the spectrum. For more details see Fig. 2(a) in [65], where the authors also compare the saddle point results to the exact results and find very good agreement.

![Figure 4.7: Direct and rescattering amplitudes for a zero-range binding potential with $U_p/\omega = 3.58$, at 800 nm and a ground state energy $I_p = 0.5$. The spectrum is in the laser polarization direction and the pulse has a $\sin^2$ envelope with zero absolute phase. The circles correspond to the ATI peaks. For the rescattering plateau, only the first five pairs of saddle points have been included.](image-url)

To assess the importance of each pair of trajectories, we plotted in Fig. 4.8 the squared amplitude for the first five pairs. The (1,2) pair’s contribution comes from the saddles whose trajectories are depicted in Fig. 4.6. This electron trajectories described by this pair have the smallest travel time. The cutoff for this pair is about $10U_p$ and this is the only pair contributing for the end
of the high energy plateau. The other pairs have smaller cutoffs and they contribute to different energy intervals with a decreasing magnitude. Note that the interference pattern for each pair has a different characteristic scale and that in the total contribution this interference effects average over, giving a much smoother interference pattern.

The last pair considered here is (9,10) and its saddles trajectories are shown in Fig. 4.6. Because of the longer travel time for this pair, the contribution to the rescattering plateau is the smallest compared to the other pairs.

![Figure 4.8: Scattering contributions for the first five saddle point pairs. The physical parameters are the same as in Fig. 4.7. The contributions have been displaced for visual convenience. The red curve shows the contribution of all the five pairs.](image)

The Stokes phenomenon is well represented for the (1,2) pair if one looks at the high energy part, around $10U_p$. If before the cutoff there is still an interference pattern given by the two saddles contributing together to the pair amplitude, after the cutoff one of the saddles has to be dropped, and the contribution shows a smooth decay, as it originates now from only one saddle.
4.3.5 Direct- and rescattered-electrons interference

For linear polarization, in the tunneling regime there is a short energy interval where direct and rescattered electrons are emitted with comparable yields and hence can interfere with significant contrast. For elliptical polarization, the yield of rescattered electrons decreases faster while the yield of the direct electrons is less dependent on ellipticity. Consequently, the energy range for interfering can be broader. For more details, see the review of Kopold and Becker [79]. The interferences have been observed experimentally [80].

4.4 Conclusions

This chapter gives a brief introduction to the theoretical treatment of Above Threshold Ionization (ATI) within the Strong Field Approximation (SFA) model. The relevant approaches along these lines encountered in the literature are presented and illustrated with specific study cases.

We focus separately on describing the direct ionization both quantum-mechanically and classically. The classical description relies on the Simpleman’s model and its success in developing an intuitive physical picture is underlined and sustained with experimental proofs. Especially, it predicts the cutoff for the kinetic energy of an ejected electron. The quantum description belongs to the widely used Keldysh-type expressions. Here, we show a particular version of it and a more general discussion is presented in the next chapter. Although we analyze the case of a stationary laser field, many features extend also to finite pulses, like the interference patterns due to beating among the different quantum trajectories.

The second part of the present review concerns the phenomenon of rescattering of the electron off the atomic core. The quantum treatment is an extension of the Keldysh-like amplitude by taking into account one more interaction of the ejected electron with the core. This is shown to explain the rescattering
plateau in the ejected electron energy spectra, seen in experiments.

The rescattering amplitude can be approximated by using the saddle point method. This leads formally to a description in terms of complex quantum orbits, much similar to Feynman’s path integral. It seems however that there is a lot of physical reality associated with these trajectories. Their real parts are almost identical to the real classical trajectories of the Simpleman’s model. The predictive power of this model is remarkable. For example, it has been demonstrated in the case of two-color harmonic generation by comparing the results from an integration of time-dependent Schrödinger equation with the Simpleman’s model [81]. The return times predicted by the Simpleman’s model exactly reflects in the temporal structure of the harmonic spectrum. The same kind of agreement holds also in the case of ATI in the way that one can identify groups of electron trajectories contributing to the ionization spectrum in a certain energy range.

Another advantage of the quantum orbits approach is that it allows for quick quantitative results, unlike solving the Schrödinger equation numerically, which for high intensities can reach today’s computational limits. To be able to run quick simulations means that one can study a broad domain of intensities or laser frequencies and draw conclusions for further research or analysis of experiments.

The biggest advantage of the quantum orbits description is that it allows us to describe an otherwise difficult phenomenon in terms of a few electron trajectories. The interferences between these trajectories are responsible for the interference patterns in the ATI spectra, observed experimentally. An important conclusion is that manipulation of quantum orbits opens the gate to control spectral features.