Appendix C

Derivation of the Bogoliubov-de Gennes equations

Here we present a derivation of the Bogoliubov-de Gennes equations for the dilute Bose gas, describing the spectrum of small-scale excitations on the background state.

If $\phi_0(\mathbf{r})$ is the time-independent background state, then weak perturbations of the wavefunction can be expressed with the trial wavefunction,

$$\psi(\mathbf{r},t) = e^{-i\mu t/\hbar}\left[\phi_0(\mathbf{r}) + u(\mathbf{r})e^{-i\omega t} + v^* e^{i\omega t}\right], \tag{C.1}$$

where μ is the chemical potential of the system. The perturbation is time-dependent with characteristic frequency ω. The amplitude functions $u(\mathbf{r})$ and $v(\mathbf{r})$ are complex. For simplicity, we will drop the time and space notation in the parameters.

The time-dependent Gross-Pitaevskii equation is,

$$i\hbar\frac{\partial\psi}{\partial t} = \left[H_0 + g|\psi|^2\right]\psi, \tag{C.2}$$

where H_0 is the single particle hamiltonian $H_0 = -(\hbar^2/2m)\nabla^2 + V_{\text{ext}}$, with V_{ext} being the external potential acting on the system, and g is the nonlinear coefficient given by equation (2.7).

We proceed by evaluating the Gross-Pitaevskii equation for the trial wavefunction of equation (C.1) to first order in u and v. The left-hand side of the

Gross-Pitaevskii equation becomes,

$$i\hbar\frac{\partial\psi}{\partial t} = e^{-i\mu t/\hbar}\left[\mu\phi_0 + (\mu+\hbar\omega)ue^{-i\omega t)} + (\mu-\hbar\omega)v^*e^{i\omega t}\right]. \tag{C.3}$$

We now evaluate the right-hand side of the Gross-Pitaevskii equation. First we consider the term $\psi^*\psi\psi$. Keeping only the terms which are linear in u and v gives,

$$\begin{aligned}\psi^*\psi\psi &= e^{-i\mu t/\hbar}(\phi_0^* + u^*e^{i\omega t} + ve^{-i\omega t})(\phi_0 + ue^{-i\omega t} + v^*e^{i\omega t})^2 \\ &= e^{-i\mu t/\hbar}\left[|\phi_0|^2\left(\phi_0 + 2ue^{-i\omega t} + 2v^*e^{i\omega t}\right) + \phi_0^2\left(u^*e^{i\omega t} + ve^{-i\omega t}\right)\right].\end{aligned} \tag{C.4}$$

Substituting all terms into the Gross-Pitaevskii equation and equating terms in $e^{-i\mu t/\hbar}$ leads to,

$$\mu\phi_0 = H_0\phi_0 + g|\phi_0|^2\phi_0. \tag{C.5}$$

Equating terms in $e^{-i(\mu+\hbar\omega)t/\hbar}$ gives,

$$(\mu+\hbar\omega)u = (H_0 + 2g|\phi_0|^2)u + g\phi_0^2 v, \tag{C.6}$$

and terms in $e^{-i(\mu-\hbar\omega)t/\hbar}$ gives,

$$(\mu-\hbar\omega)v^* = (H_0 + 2g|\phi_0|^2)v^* + g\phi^2 u^*. \tag{C.7}$$

Taking the complex conjugate of equation (C.7) leads to,

$$(\mu-\hbar\omega)v = (H_0 + 2g|\phi_0|^2)v + g\phi^{*2}u. \tag{C.8}$$

Equations (C.5), (C.6) and (C.8) can be written in the form of a matrix equation in terms of u and v,

$$\begin{pmatrix} L & M \\ M^* & L \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \hbar\omega\begin{pmatrix} u \\ -v \end{pmatrix}, \tag{C.9}$$

where,

$$L = H_0 - \mu + 2g|\phi_0|^2, \tag{C.10}$$

and,

$$M = g\phi_0^2. \tag{C.11}$$

This is the Bogoliubov-de Gennes equation for the dilute Bose gas. Solutions to this matrix equation can be obtained by solving the eigenvalue equation,

$$\begin{vmatrix} L-\hbar\omega & M \\ M^* & L+\hbar\omega \end{vmatrix} = 0. \tag{C.12}$$

It follows that the energy of the excitations is given by,

$$\hbar\omega = \sqrt{L^2 - |M|^2}. \tag{C.13}$$

In particular, for a homogeneous system with background density n, $V_{\text{ext}} = 0$, $H_0 = \hbar^2 k^2/2m$, and $\mu = ng$, where k is the wavevector of the excitation. The energy spectrum is then given by,

$$\hbar\omega = \sqrt{\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + 2ng\right)}. \tag{C.14}$$

This dispersion is particle-like ($\hbar\omega = \hbar^2 k^2/2m$) at high momenta, while at low momenta, the spectrum is linear ($\omega = ck$) and supports sound (phonon) waves with characteristic speed c.