Numerical Studies of Vortices and Dark Solitons in Atomic Bose-Einstein Condensates

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Abstract

Dilute atomic Bose-Einstein condensates support intriguing macroscopic excitations in the form of quantized vortices and dark solitons. In this thesis we present extensive quantitative studies of the dynamics of these phenomena in the limit of zero temperature, performed by numerical simulation of the Gross-Pitaevskii equation. We show that vortices and dark solitons are inherently unstable to acceleration through the condensate, leading to the emission of sound waves. Indeed, for a single vortex/soliton, the power emitted is proportional to the square of the local acceleration. However, due to the finite size of the condensate, the vortex/soliton reinteracts with the emitted sound. This coupling has an important effect on the ensuing dynamics, and we illustrate how it can be engineered to induce net decay, stabilise, or even drive energy into the vortex/soliton.

Under the harmonic confinement typically employed to trap the condensates, the acceleration-induced decay is stabilised by reabsorption of the emitted sound. However, by modifying the trap geometry, e.g. by adding an inner dimple in which the soliton resides or an optical lattice potential, it is possible to break this equilibrium and so induce a net decay of the vortex/soliton in a controllable manner. The decay rate can be considerable and should be observable under current experimental conditions. The dynamical stability of quantized vortices is also relevant to the field of turbulence in superfluid Helium, where the motion of the vortices is induced by the surrounding distribution of vortices rather than density. We extend these results to this field, and additionally consider the interactions between two and three vortices, which are also found to involve sound emission.

By exciting the sound field of the condensate it is possible to drive parametrically energy into a dark soliton. In a real dissipative environment, this can be used to stabilise the soliton decay. Finally, we illustrate the links between dark solitons and vortices: a dark soliton embedded in a three-dimensional system is prone to decay into vortex rings, while a vortex in a quasi-one-dimensional geometry cannot be supported and exists as a hybrid between a vortex and dark soliton, known as a solitonic vortex.
Dedicated to Mum and Dad,
for all their love, friendship, and support.
Declaration

I confirm that no part of the material offered has previously been submitted by myself for a degree in this or any other University. Where material has been generated through joint work, the work of others has been indicated.

Nicholas Parker
Durham, 7th October 2004

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